THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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PROJECTIVE GEOMETRY.

PROJECTIVE geometry derives both interest and importance from the fact that it assumes only the simplest and most fundamental axioms. The following sketch of the first principles, and proof of the fundamental theorem, is taken from Prof. Federigo Enriques' *Projektive Geometrie* (B. G. Teubner, Leipzig, 1903, 8vo, pp. xiv, 374). This work is specially adapted for school teaching. The first (Italian) edition appeared in 1898, and the axioms on which it is founded were explained four years earlier in the *Rendiconti R. Ist. Lombardo*, Vol. 27, pp. 550-567.

Projective geometry is the geometry originated by von Staudt in the *Geometrie der Lage*, of which an excellent account by Prof. C. A. Scott has already been given in the *Mathematical Gazette*, Vol. I., Nos. 19, 20, 22, pp. 307, 323, 363. Russell's very original *Foundations of Geometry* is the only English work which treats the subject from a strictly logical standpoint, but is unfortunately very unreliable as far as the mathematics is concerned. The more serious errors are tacitly abandoned in his recent book on *The Principles of Mathematics*; but a large part of this important work is very indefinite, or at least leaves no definite impressions; and its use of technical terms, without any guiding reference, is very perplexing.

Opinions differ as to how projective geometry should be defined, and to what extent its axioms should be restricted. It is usual to regard it as including all those properties and relations of figures which can be stated without reference to the conception of magnitude. There are different varieties (elliptic, parabolic, hyperbolic), but they differ much less from one another than the corresponding types of metrical geometry. Two notions seem to be essential (1) that space is divisible into elements, called points, certain derived elements being called straight lines and planes, and (2) that these elements can be arranged systematically, so that *order* (or a co-ordinate system) results from the arrangement. Order must be given either explicitly or implicitly by the axioms. A further axiom, the axiom of continuity, is essential for the development of projective geometry as at present understood, since otherwise only a very mutilated theory of double points would be possible. It has not been proved, however, that no theory of projective geometry is possible without this axiom. The axiom is also open to the objection that it appears to be based too intimately on preconceived notions of arithmetic.

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Two straight lines in a plane which do not meet in a proper point are said to have an improper or ideal point common. Ideal points can be made to correspond projectively to proper points. We shall regard all imaginaries as excluded, owing to the unsatisfactory way in which they must be introduced, and the complicated considerations to which they give rise.

Enriques, following von Staudt, regards projective geometry as visual and intuitional; and assumes the axioms of Euclid with the exception of the axiom of superposition. In place of saying that two lines are parallel (which in Euclid involves a metrical notion) he says that they have an improper or ideal point common. Thus every straight line contains one and only one ideal point, which has the effect of *closing* the line. Two points A, B on a line divide it into two segments; and if C is in one segment, and D in the other, the pairs A, B and C, D are said to separate one another. A plane contains *one* ideal line (we omit the prefix *straight*), and space contains *one* ideal plane. The elements of space, viz. points, lines, and planes, are given intuitionally. In space of three dimensions either points or planes may be regarded as the original elements; but in both cases lines are derived elements. Forms of three grades are distinguished; in the first grade are the series or range of all points on a line, the flat pencil of all lines in a plane through a point, and the axial pencil of all planes through a line, the point and line being called the vertex and axis of the two pencils respectively. Forms of the second grade include the point-system of a plane, the line-system of a plane, the sheaf of all lines through a point, and the sheaf of all planes through a point. The forms of the third grade are the point-system of space and the plane-system of space. The line-system of space is of a higher grade, giving rise to line geometry. Two forms are perspective if both are sections of the same form, or if one is a section of the other, such as an axial pencil and its section by a line or plane. Two forms are projective if it is possible to pass from one to the other by means of a finite number of perspectivities. For a full discussion of these conceptions and definitions see Prof. Scott's articles in the *Mathematical Gazette* referred to above.

Enriques divides the axioms into two sets of three each. The first set presents no distinctive peculiarity, and states no property which has not been generally admitted as axiomatic. The first set of axioms is as follows:—

I. In a form of the third grade two original elements determine a like form of the first grade, to which the two elements belong.

It is of course understood that the form determined by the two elements is contained in the given form of the third grade. A similar remark applies to axioms II. and III. The elements include *all* the elements, both proper and ideal.

Thus (a) two points determine *one* straight line, or range of points, to which they belong; and (b) two planes determine *one* straight line, or axial pencil, to which they belong.

II. In a form of the third grade three original elements, which do not belong to a form of the first grade, determine a like form of the second grade, to which the three elements belong.

Thus (c) three points, not on a straight line, determine (the point-system of) *one* plane; and (d) three planes, not through a common line, determine *one* sheaf of planes.

III. In a form of the third grade an original element and a form of the first grade, which do not belong to one another, determine a form of the second grade, to which they belong.

Thus (e) a point and range determine *one* plane; and (f) a plane and axial pencil determine *one* sheaf of planes.

From these axioms it also follows that in a form of the second grade two forms of the first grade have always *one* element common. Also von Staudt's theorem concerning harmonic sets of four elements follows without

any further axioms. Before considering this, however, it will be more convenient to give Enriques' remaining axioms. These are more difficult to decide upon, and also more debatable, than the first set of axioms given above. The principal part of the fourth axiom is as follows :—

IV. The elements of a form of the first grade possess a natural cyclical order, which may be taken in the one or the opposite sense. Any element may be taken as the first element; but between any two elements there are an infinite number of other elements.

Four elements can be divided into two pairs which separate one another in only one way. Thus if they occur in the order A, B, C, D then A and C separate B and D. Also two elements divide a form of the first grade into two parts or segments.

V. If two like forms of the first grade are perspective and an element describes a segment AB of the one, the corresponding element describes a segment $A'B'$ of the other, where A' corresponds to A and B' to B.

To every element X in one form corresponds *one* element X' in the other, by the first set of axioms; and so long as X lies in one particular segment of the first form, axiom V. states that X' lies in a particular segment of the other, and when X goes over to the other segment so does X' go over to the other.

Axioms IV. and V. are capable of analysis, and can be reduced to others which are not so comprehensive; but, regarded as intuitive axioms, nothing could be simpler, or better adapted for elementary teaching. There remains only the axiom of continuity, or Dedekind's axiom.

VI. If a segment AB of a form of the first grade is divided into two parts, so that—

(1) each element of the segment AB belongs to one part or the other,

(2) A belongs to the first part and B to the second,

(3) each element of the first part is before each element of the second part;

then there exists *one* element C of the segment AB (which may belong to either part) such that each element of AB which goes before C belongs to the first part, and each element of AB which comes after C belongs to the second part.

We now come to the well-known theorem, which von Staudt showed to be a consequence of the first set of axioms, that in a form of the first grade to three given elements corresponds a fourth unique element forming with the three a harmonic set. It will be sufficient to prove this for the range of points on a line. Let E, H, F be the three given points (see figure next page). On a line through H take any two points A, C; let EA, FC meet in B, and EC, FA in D; then the point K where BD meets EF is the fourth unique element. H, K are called a harmonic pair with respect to E, F, and may be described as the points in which EF, regarded as a diagonal of any quadrilateral $ABCD$, is cut by the other two diagonals.

To prove that K is unique, draw any other line $HA'C'$ through H, not lying in the same plane as HAC and EF, and construct the quadrilateral $A'B'C'D'$ and the point K' corresponding to H, A', C'. Then AA' and BB' lie in the plane containing the two lines EAB and $EA'B'$; hence AA' and BB' meet one another, similarly BB' and CC' meet, and CC' and AA' meet; therefore AA', BB', CC' meet in one and the same point, for they do not lie in one plane, since the planes ABC, $A'B'C'$ are different by hypothesis.

By a similar proof DD' passes through the point where AA' and CC' meet, and therefore meets BB'; hence BD and $B'D'$ meet, and both meet EF (in K and K'); therefore BD, $B'D'$, EF, not lying in one plane, meet in one and the same point, *i.e.* K' coincides with K.

Also, if we construct the point K'' for any quadrilateral in the same plane as $ABCD$, having EF for a diagonal, then K'' coincides with K', and therefore with K; hence K is unique, Q.E.D.

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To prove that K is unique, draw any other line $HA'C'$ through H, not lying in the same plane as HAC and EF, and construct the quadrilateral $A'B'C'D'$ and the point K' corresponding to H, A', C'. Then AA' and BB' lie in the plane containing the two lines EAB and $EA'B'$; hence AA' and BB' meet one another, similarly BB' and CC' meet, and CC' and AA' meet; therefore AA', BB', CC' meet in one and the same point, for they do not lie in one plane, since the planes ABC, $A'B'C'$ are different by hypothesis.

By a similar proof DD' passes through the point where AA' and CC' meet, and therefore meets BB'; hence BD and $B'D'$ meet, and both meet EF (in K and K'); therefore BD, $B'D'$, EF, not lying in one plane, meet in one and the same point, *i.e.* K' coincides with K.

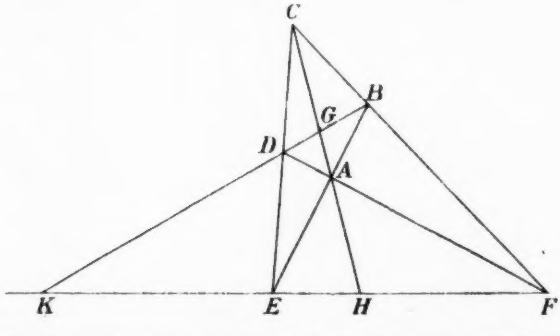
Also, if we construct the point K'' for any quadrilateral in the same plane as $ABCD$, having EF for a diagonal, then K'' coincides with K', and therefore with K; hence K is unique, Q.E.D.

The above proof would fail if we were restricted to geometry of two dimensions. In this case either points or straight lines could be regarded as original or fundamental elements; and there would be only one axiom corresponding to those of the first set above, viz.:—In a form of the second grade two original elements determine a form of the first grade, to which they belong. It would seem that this axiom would have to be supplemented by others in order to prove the uniqueness of the fourth element of a harmonic set. In two dimensions there is no set of elements, like straight lines in three dimensions, which cannot be regarded as original elements.

In two perspective ranges a harmonic set always corresponds to a harmonic set; and the same is therefore true for any two projective ranges. This is seen by constructing any quadrilateral for one harmonic set, and considering the corresponding perspective quadrilateral.

If H, K is a harmonic pair with respect to E, F it is clear that the relation between H, K is a reciprocal one; so also is the relation between the two pairs H, K and E, F, *i.e.* E, F is a harmonic pair with respect to H, K. In the figure let FG meet AB in A' and CD in C', and let EG meet BC in B' and DA in D'. From the quadrilateral $GB'CC'$ we see that $B'C'$ passes through K; and similarly from the quadrilaterals $GD'AA'$, $GA'BB'$, $GC'DD'$ we see that $D'A'$ passes through K, and $A'B'$ and $C'D'$ pass through H. Hence HK is a diagonal of the quadrilateral $A'B'C'D'$, and the other two diagonals meet HK in E, F, which proves the theorem. The triangle GHK formed by the diagonals of the quadrilateral $ABCD$ is its harmonic triangle, and is also the harmonic triangle of the quadrangle $A'B'C'D'$; EFG is the harmonic triangle of the quadrilateral $A'B'C'D'$ and of the quadrangle $ABCD$.

The next important theorem which we have to consider is the following:—Any harmonic pair separates the pair with respect to which it is harmonic, *i.e.*, the harmonic pairs of any harmonic set are the two pairs which separate one another. This is proved by means of axioms IV. and V.



In the figure we have to prove that H, K separate E, F. Take a point P in EAB such that E, P, A, B occur in order; let DP cut HAC in O and EF in X, and let BO cut ECD in Q and EF in Y. Then projecting E, P, A, B from D on to EF, we see that E, X, F, K occur in order, or X lies in the opposite segment EF to that in which K lies. Projecting E, P, A, B form O on to ECD, we have E, D, C, Q, and projecting these from B on to EF, we have E, K, F, Y; therefore K, Y lie in opposite segments EF, *i.e.* X, Y lie in the same segment EF, and K in the opposite one. Finally project E, P, A, B from O on to EF, and we have E, X, H, Y. Hence in considering the order E, X, F we have E, X, H, Y, F or E, Y, H, X, F in

order, according as X is before or after Y; *i.e.* H is in the segment EF which contains X, Y, and does not contain K, Q.E.D.

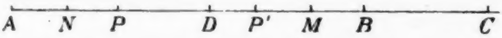
The *fundamental theorem* is the following:—In two projective forms of the first grade any three elements of one may correspond to any three elements of the other; and then to any fourth element of one corresponds a unique element of the other. We shall prove this as before for ranges of points only.

(1) Let A, A', B', B be four points in order on a line, and let AB, $A'B'$ be corresponding segments of two projective ranges, so that, as a point describes the segment AB in the sense $AA'B$ the corresponding point describes $A'B'$ in the same sense. Then there exists a double point M within the segment $A'B'$, such that there is no double point in $AA'M$ before M.

The segment AB can be divided into two parts, of which the first part contains all points X possessing the property that each and every point in the segment AX goes before its corresponding point, and the second part contains all points Y possessing the property that some point exists in AY (which may be Y itself) which does not go before its corresponding point. The points X, Y make up all the points of the segment A, B; also A is a point X and B is a point Y; also every X goes before every Y; hence (axiom VI.) there is a point M in the segment $AA'B$ such that every point before M is an X, and every point after M is a Y. Also, since the correspondent of any point in AB is in $A'B'$, it is clear that every point in AA' (excluding A') is an X, and every point in $B'B$ a Y. Hence M is somewhere in $A'B'$.

Let M' be the point which corresponds to M. Then, if M' is before M, any point within $M'M$ is an X, whose correspondent X' is after X, and therefore after M'. But since X is before M, X' is before M' (axiom V.); therefore X' is both before and after M', which is impossible. Again, if M' is after M, any point within MM' is a Y, whose correspondent Y' is after M' (because Y is after M), and therefore after Y. Thus Y is an X, by definition, which is impossible. Hence M' is neither before nor after M, and must coincide with M, *i.e.* M is a double point. Also, there is no double point in $AA'M$ before M, since every point before M is an X.

(2) If two projective ranges on the same line have three double points, then every point is a double point, *i.e.* the projectivity is identical (fundamental theorem of von Staudt).



Let A, B, C be the three double points, and suppose that the projectivity is not identical, so that there is some point P whose correspondent P' does not coincide with P.

We may assume that P does not lie in the segment ACB; for if A', B', C' are three points such that A, A' separate B, C and B, B' separate C, A and C, C' separate A, B, then A, C', B, A', C, B' occur in order; and P must lie in one of the segments $AC'B$, $BA'C$, $CB'A$, and we may suppose it to lie in $AC'B$. The segment ACB corresponds to itself, since C corresponds to itself; hence APB also corresponds to itself, and therefore P' lies in APB. We may also assume that A, P, P', B occur in order, by interchanging A and B if they are not in order.

The segment $P'B$ corresponds to PB, both taken in the sense APB. Hence, by (1), there is a double point M in $P'B$, which may be B, such that there is no other double point in PM. The reasoning of (1) still holds when the corresponding points B B' coincide, as is the case here. Again, reversing the

order and considering the reciprocal projectivity, to the segment $P'A$ corresponds the segment PA, both in the sense BPA. Hence there is a double point N in PA, which may be A, such that no other double point exists in $P'N$. We have thus found two double points N, M, such that no other double point exists in the segment NPM.

Take the point D such that C, D and M, N are harmonic pairs. Then since any set corresponding to a harmonic set is harmonic, and each of the points C, M, N corresponds to itself, D can only correspond to itself. Hence D is a double point within the segment NPM, which is impossible from above. Hence a projectivity which contains three double points must be identical.

(3) In two projective ranges any three points of one may correspond to any three points of the other.

When the ranges are on two lines not in the same plane any three points of one A, B, C are perspective with any three points A', B', C' of the other. Take any point L on AA', other than A or A', and let the plane LCC' meet BB' in M; then LM meets CC' in a point N. Then considering the axial pencil with LMN as axis, it is clear that the three planes of the pencil which meet ABC in A; B, C will meet $A'B'C'$ in A', B', C' respectively; *i.e.* A, B, C are perspective with A', B', C'.

If the lines are the same, or in the same plane, take any three points A'', B'', C'' on a line which is not in a plane with either ABC or $A'B'C'$. Then A, B, C and A', B', C' are both perspective with A'', B'', C'', and therefore projective with one another.

Finally if A, B, C, D on one line are projective with A', B', C', D' on another (or the same), and A, B, C, D, A', B', C' are fixed, then D' is fixed; for if A, B, C, D are also projective with A', B', C', D'', then A', B', C', D' are projective with A', B', C' D'', and therefore D'' coincides with D', by (2). This completes the proof of the fundamental theorem for projective ranges; and, in the same way, the theorem may be proved for any two projective forms of the first grade.

We have only referred to a small portion of Enriques' *Projektive Geometrie*, but we have given sufficient to show that it is a book of more than ordinary interest. The whole work is a very complete one, and includes the theory of conics and of projectivity in forms of all the three grades. In another article we may consider the projective coordinate system, which does not come within the scope of Enriques' book.

F. S. MACAULAY.

MATHEMATICAL NOTES.

142. [K. 20. d.]. *To prove geometrically the principal trigonometrical relations of two angles.*

Let O be the centre, OX a radius of a circle whose diameter $=1$.

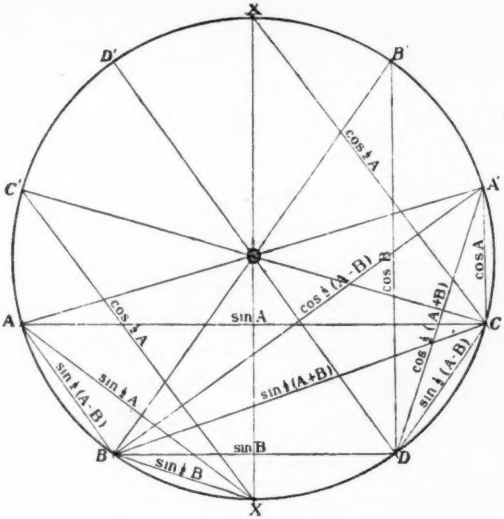
Draw the radii OA, OC making angle A on each side of OX.

Similarly the radii OB, OD making angle B.

Let X', A', B', etc. be the extremities of diameters from X, A, B, etc.

hen we can write down

$\sin A = AC = A'C'$.	$\cos A = AC' = A'C$.
$\sin B = BD = B'D'$.	$\cos B = BD' = B'D$.
$\sin\frac{A}{2} = AX = CX$.	$\cos\frac{A}{2} = AX' = C'X$.
$\sin\frac{B}{2} = DX = BX$.	$\cos\frac{B}{2} = BX' = DX'$.
$\sin\frac{A+B}{2} = AD = BC$.	$\cos\frac{A+B}{2} = AD' = BC'$.
$\sin\frac{A-B}{2} = AB = CD$.	$\cos\frac{A-B}{2} = A'B = CD'$.



Now apply Euclid VI. D. to different quadrilaterals in turn, and we write down the required relations at sight :

Quadrilateral	*gives*
$AXCX'$	$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$.
$BXCX'$	$\sin\frac{A+B}{2} = \sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}$.
$ABXX'$	$\sin\frac{A-B}{2} = \sin\frac{A}{2}\cos\frac{B}{2} - \cos\frac{A}{2}\sin\frac{B}{2}$.
$ACBD$	$\sin A\sin B = \sin^2\frac{A+B}{2} - \sin^2\frac{A-B}{2}$.
$ABCB'$	$\sin A = \sin\frac{A+B}{2}\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\sin\frac{A-B}{2}$.
$AA'BD$	$\sin B = \sin\frac{A+B}{2}\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\sin\frac{A-B}{2}$.
$ADCB'$	$\sin A\cos B = \sin\frac{A+B}{2}\cos\frac{A+B}{2} + \sin\frac{A-B}{2}\cos\frac{A-B}{2}$.

and so on. W. A. WHITWORTH.

107. [P. 3. B.] *Continued Inversion by Coaxal Circles.*

Mr. R. F. Davis's Note (p. 383) on Double Inversion suggests some curious consequences. Remarking that his three circles of inversion (A), (B), (E), are all orthogonal to PQR, and are therefore coaxal ; and that the bisector of EF, being a diameter of PQR, is their radical axis ; we may state his theorem thus :

Successive inversion by any two circles $(A)(B)$ is equivalent to inversion by the coaxal circle (E) followed by reflection in the radical axis (R); or to reflection in (R) followed by inversion by the coaxal circle (F).

Or briefly, in symbols,

$$[AB]=[ER]=[RF].$$

The points B, E are inverse for (A), and A, F are inverse for (B). When (A), (B) are orthogonal, E and F coincide, and $[AB]=[BA]$.

(1) Take (C) any other circle coaxal with (A), (B), and let F, D be inverse points for (C). Then

$$[ABC]=[RFC]=[RRD]=[D].$$

And evidently we can carry on this process to any extent, till we get the result:

Successive inversion by any odd number $(A)(B)(C)\dots(K)$ of a series of coaxal circles amounts only to a simple inversion by that one of the series whose centre is where the point A arrives after inversion by $(B)(C)\dots(K)$.

Another way of finding this centre would be to perform the operation $[ABC\dots K]$ on a point P, obtaining Q, R, ... P', (which are all concyclic), and to draw PP'.

(2) $[ABCABC]=[DD]=1$; that is, the six successive inversions produce no change.

(3) Let (C) be the circle into which (B) inverts (A). Then, choosing P on the circumference of (A), we find that D coincides with B; $\therefore$ $[ABC]=[B]$. This result includes two others, well known: (a) when a circle (A) is inverted into another (C) the inverse of the centre of (A) is with respect to (C) the inverse of the centre of inversion: and (b) when a circle (A) and two points inverse for (A) are inverted, they become a circle (C) and two points inverse for (C).

(4) Let $(A)(B)(C)$ be a set of circles each of which inverts the other two into each other. Then, by (3),

$$[AB]=[AABC]=[BC].$$

Similarly $[BC]=[CA]$; and $[AC]=[CB]=[BA]$. Hence:

If any point P be taken, and its three inverses for $(A)(B)(C)$, and of these again all the possible inverses, and so on to any extent, no more than five new points will be obtained; namely, those which result from the operations $[A]$, $[B]$, $[C]$, $[AB]$, $[AC]$; for all other operations, however long and varied, can be reduced to one of these.

These circles $(A)(B)(C)$, if real, must have common points O, O'; otherwise each circumference would have to pass between the other two. And of the radii OA, OB, OC, each bisects the angle between the other two, because for instance $AB:AC=OB:OC$; hence the angles of intersection are all 60° or 120°. The three Apollonian circles of a triangle have this property.

C. E. Youngman.

REVIEWS.

The Algebra of Invariants. By J. H. Grace, M.A., and A. Young, M.A.

The Theory of Invariants, since it was commenced by Boole in 1844, has shewn, now for nearly sixty years, a continuous development. It has given rise to several entirely new methods, which have in succession enabled fresh discoveries to be made. When, for example, Cayley's method of generating functions had been pushed to its furthest limits, and had become indeed a fruitful source of error, the symbolic treatment in the hands of Aronhold, Clebsch, Gordan, and others, was successful to an extraordinary degree, not only in showing how the errors arose, but also in establishing the facts. No one perhaps has imported more new ideas into the subject than Sylvester, and here also Hammond Franklin and others enabled

progress when a block upon the line looked inevitable. When the theory began it was entirely algebraical, and limited in its scope to multinomial quantics. Its proper place in mathematical theory as a principle connected with the theory of groups was determined by Sophus Lie, Klein, Poincaré, and other workers on the continent of Europe. In particular the invariance of differential and functional expressions in respect of certain groups of substitutions finite or infinite, continuous or discontinuous, was studied and foundations solidly laid, principally by Sophus Lie. In Great Britain, what may be called the advanced work in the theory is almost entirely due to Forsyth, who has treated the differential invariants connected with algebraic curves and surfaces with success.

The work under review is entirely concerned with the Invariants Theory of Quantics, a theory which, it should be said, is of fundamental importance in regard to the more general theory. Most of the invariants theories that have been studied, including Sylvester's theory of Reciprocants, and those recently treated by Forsyth, depend in large measure upon the Invariant Theory of Quantics; the same partial differential equations and the same fundamental forms crop up time after time, and the same notions and methods are found effective in making progress. Original ideas and successful methods are available over the whole field.

Previous to the appearance of the present book, the only works of note on the same subject in the English language were Salmon's *Higher Algebra* and Elliott's *Invariant Theory of Quantics*. The former is a resumé of the work of Cayley, of Sylvester, and of the School of Trinity College, Dublin, with a slight sketch only of the symbolic method. The latter is a fairly exhaustive account of the progress that has been made by means of differential equations and operations and generating functions, to which its author has largely contributed; it does not touch on symbolic treatment.

We have here what the authors modestly describe as "an English introduction to the symbolical method." It does not pretend to be an exhaustive treatise—a series of volumes would be necessary for that—but an attempt to give sound instruction and information to students who wish to study the classical memoirs and themselves take part in research. In this it is quite successful, for it shews step by step how the authors have, within the last years, made discoveries of the greatest importance entirely by the symbolic method.

After an elementary exposition in the first five chapters we are given an account of Gordan's theorem, and the connected work of Jordan, Hilbert, and Stroh. The main object is to prove that for any system of quantics there exists an arithmetical system of invariants containing a finite number of forms, of which all other invariants are rational integral functions. In distinction to this there exists an algebraic system of forms finite in number, of which all other invariants are algebraic functions; this is a less difficult theory, on which the chief authority is Forsyth, who, in masterly memoirs in the *American Journal of Mathematics*, has practically made it complete by applying the theory of linear partial differential equations. Many mathematicians are of opinion that there is no theoretical advantage in obtaining an arithmetical system, and that its determination is of sentimental interest. Be that as it may, there is no doubt that the search after the arithmetical system has given birth to refined and beautiful mathematical work, which has had a large measure of success. The authors base the proof that the number of irreducible forms is finite upon a property of Diophantine Equations that was first brought into prominence by Hilbert; the theorem states that "the number of irreducible solutions in positive integers of a system of homogeneous linear equations is finite." The authors, however, fail to notice the most important fact in regard to such a system, viz., that it is invariably associated with a syzygetic theory; it is this circumstance that has enabled the construction in several instances of a syzygetic theory of ground forms appertaining to groups of transformations. It would have been of much advantage to students of this book if syzygetic theories for some simple Diophantine systems had been given in detail, for although their existence is implied by the Lemmas of Hilbert, the real point is likely to be missed in the absence of an explicit statement with illustrations. The manner in which the Lemmas in question are set forth in Chapter IX. otherwise leaves nothing to be desired.

The tenth and eleventh chapters give a satisfactory account of the geometrical interpretation of binary forms and of the theory of "apolarity." For references in regard to the latter theory the reader is pointed to Meyer's *Berichte*, an exceed-

ingly valuable work that should be in the hands of every student of invariant theory. After an account of the symbolic method of treatment of ternary and quaternary forms we come to the most novel and important part of the treatise, the chapters on "Types" and "General Theorems on Quantics." The authors here are on the frontiers of knowledge, and make some epoch-making discoveries. By a consideration of the covariant types of a system of binary quantics, of which none of the orders exceeds n, they determine the maximum order of an irreducible covariant of such a system. If $n=2\lambda+n_1$, the order in question is shewn to be

$$(\lambda-1)2\lambda+n_1(\lambda+1)+2,$$

and the further observation is made that if the results for perpetuants are absolutely accurate (and of this there is practically no doubt) the maximum order is always reached, even for a single quantic of order n, except for the case $n=3$. This beautiful result is one of the most important ever made in the theory of binary forms, and would give distinction to the book if it contained nothing else of value. In Chapter XVI., moreover, is given an account of the "quantitative substitutional analysis," a powerful method recently invented by Mr. Young. It is applied to prove in a single page the Theorem of Peano, which asserts that "every type of the binary n-ic which does not furnish an irreducible covariant for a system of n n-ics is reducible, with one possible exception." Similar results are obtained for forms in any number of variables; thus, for instance, it is shewn that all covariants of any number of ternary n-ics may be obtained by means of Aronhold operators from the system for $\binom{n+2}{2}-1$ ternary n-ics, with one possible exception. The substitutional analysis of Mr. Young is an operative instrument based upon the theory of groups, which is sure to have an important future in connection with modern algebra; the authors have clearly explained it, and placed in the hands of students a weapon of research more useful, I believe, than any other that has been invented up to the present time for effective work in this part of algebra.

A book of less than 400 pages cannot possibly cover more than a small portion of the subject matter of invariant theory; it is therefore not surprising to find that the subject of the invariants of the orthogonal substitution is not touched upon. The methods of the authors are singularly well adapted to their consideration, which perhaps some day they will take up.

No better book can be placed in the hands of the rising generation of mathematicians.

Feb. 24, 1904. P. A. MacMahon.

Théorie Nouvelle des Fonctions, exclusivement fondée sur l'idée de nombre. Gustave Robin. **Oeuvres scientifiques réunies et publiées** par Louis Raffy. (Gauthier-Villars.) Paris, 1903.

The present volume of some 200 pages contains the work in pure mathematics of Gustave Robin, who has hitherto been known as the author of a number of memoirs on applied mathematics and physics which are to form two subsequent volumes of the present edition of his works.

We learn that the theory now presented was elaborated during the last years of Robin's life. The fundamental idea had occurred to both author and editor when they were students together: it dominated the course on the elements of analysis which Robin gave in 1892-1893 to students in physics at the Faculté des Sciences at Paris.

Sensitive and diffident, Robin never published his work. In fact, the editor says in his preface: "*Sachant qu'il ne consentirait jamais à l'écrire lui-même, je le décidai, pendant les vacances de* 1894, *à m'en exposer chaque jour, suivant ses reflexions du moment, telle ou telle partie: le tout fut pris en notes. C'est à l'aide de ces notes qu'a été établi le texte de la Théorie nouvelle des fonctions.*"

A volume so written must be necessarily fragmentary: lacunae must occur which we should wish to see filled up: we expect to find developments omitted which belong to the main theory. The care of the editor has, however, given us a theory which, so far as it is independent of the theory of functions of two independent variables and all the subsidiary integral investigations associated with the name of de la Vallée-Poussin, contains most of the results which we can legitimately demand.

But at the outset the reader will undoubtedly feel a sense of dismay. In the preface we read that MM. Robin and Raffy desired to build up a theory from which "the so-called (*prétendus*) irrational numbers, pure symbols of incommensurable geometrical magnitudes, should be excluded." In the introduction we are told that the authors "will put forward the reasons which make us reject from our science, some as useless, some as unjustifiable, the ideas and developments which occupy the central place in modern treatises on analysis." Finally, towards the end of the introduction: "By the admission of those very authors who make the theory of the so-called irrational numbers the basis of analysis, there are no *numbers* to which we can give such a name; and when, for the basis of this theory, we say that certain operations define an irrational number, we express nothing. We are only guilty of an abuse of language in that we forget the meaning of the word *define*." And once more: "The traditional confusion between numbers and magnitudes is the vice of modern mathematics." "Modern analysis is *un jeu stérile de l'esprit.*" Thus the three great presentations of the theory of irrational number, associated respectively with the names of Weierstrass and Pincherle, Cantor and Heine, and Dedekind, are to be replaced by the ideas which MM. Robin and Raffy put forward. The authors certainly do not lack boldness: some of the statements which have been quoted make us turn to the *imprimatur* of the title-page, 'sous les auspices du ministère de l'instruction publique,' to be quite sure that we have not to deal with that scientific nuisance, 'the crank.'

But when we turn to the theory proper, there is no development of the controversial attitude of the introduction. The position taken up is stated clearly, the definitions are lucid, and successive propositions are built into the general theory with elaborate care.

According to the fundamental idea, $\sqrt{2}$ is the measure of a geometrical magnitude, the ratio of the diagonal to the side of a square. This is a concept not derived from the idea of pure number. If we restrict ourselves to the latter class of ideas, we must consider the decimal fraction 1·4142 ....

This fraction, if we stop at any decimal place, is a rational number. The sequence of such rational numbers is such that, after a time, any one means the same as any succeeding one to our finite senses. The difference, for instance, between the fraction taken to the 1,000,000$^{\text{th}}$ and that to the 1,000,001$^{\text{th}}$ decimal place cannot be measured by these senses. If then we have a real convergent sequence

$$a_1, a_2, \dots, a_n, \dots,$$

which does not tend to a rational limit, we may say that it defines a number a, where a means any term of the sequence after the nth, n being sufficiently large. The sequence is termed an *irrational sequence*, and the interval between two numbers a and b, so defined, is termed an *irrational interval*.

If to each number which belongs to any interval (a, b) we make correspond a known convergent sequence, we say that the aggregate of all these sequences *defines a function* of the variable x in the interval (a, b). Thus we make our function correspond to each rational number x, but not to each convergent sequence $x_1, \dots, x_n, \dots$, between a and b. For if the latter sequence were irrational and $f(x)$ designated the number corresponding to it, we should mean by x no one of the terms of this sequence nor even any (rational) number. The numbers x between a and b for which $f(x)$ is defined do not therefore form a *continuum*.

Such are the fundamental differences between M. Robin's theory and those theories which have become classic. The first startling result to which the new theory gives rise is that *a function, finite within any interval, does not in general attain its upper and lower limits.* Thus $1-(x^2-2)^2$ never attains its upper limit unity.

The ideas of mean oscillation and integrable functions, and the condition for integrability, accord in the main with the classic theories.

The definition of uniform continuity is stated as follows:—*$f(x)$ is uniformly continuous in the interval* (x_0, X) if, given a positive fraction ϵ as small as we please, we can determine a number α, such that for each pair of numbers x and x', which belong to the interval and differ by less than α,

$$|f(x')-f(x)|<\epsilon,$$

where by $f(x')$ and $f(x)$ we mean two terms (each of sufficiently high order) of

the convergent sequences which define $f(x')$ and $f(x)$. On the other hand $f(x)$ *is continuous for the value* x if we can calculate a positive number α such that

$$|f(x+h)-f(x)|<\epsilon, \text{ when } |h|<\alpha.$$

As a result *a function can be continuous for every number* x *belonging to an interval without resting finite in this interval.* (There appears to be an error in the explanation of the example given.)

In Chapter V. the idea of differentiation is introduced. Let $k=f(x+h)-f(x)$: then if the sequence $\frac{k}{h}, \frac{k'}{h'}, \frac{k''}{h''}, \ldots$, in which the denominators tend to zero, is convergent and equal to every analogous sequence, we call each term of the sequence of sufficiently high order the *derivate* of $f(x)$ and denote it by $f'(x)$.

From this definition arises the theorem that *if the derivate of a function is zero for all values of* x *within an interval, the function is* **not** *necessarily constant in this interval.* And it is possible to construct functions which are *continuous* in every interval, which cannot be decomposed into a definite number of sub-intervals where they are constant, and which have nevertheless a derivate zero throughout the whole interval.

We have to introduce the idea of uniform differentiability to obtain propositions analogous to those of the classic theories: $f(x)$ is *uniformly differentiable* in an interval (a, b) if for any fraction ϵ we can determine a number α such that

$$\left|\frac{f(x')-f(x)}{x'-x}-f'(x)\right|<\epsilon, \text{ so long as } |x'-x|<\alpha,$$

x' and x lying within the interval.

When a function possesses a derivate but is **not** uniformly differentiable, α, instead of being constant, depends upon x. As an example,

$$f(x)=x^2\sin\frac{1}{x}\,[f(0)=0]$$

is not uniformly differentiable in an interval including $x=0$, although its derivate is everywhere finite.

A function $f(x)$ which is uniformly differentiable within an interval (a, b) is continuous in this interval, as is also its derivate. If the function vanishes for a and b, its derivate vanishes within the interval. The first theorem of mean value holds for such a function, and for uniformly differentiable functions the derived set of propositions accords with the usual theory.

But the first theorem of mean value holds for other functions than those which are uniformly differentiable, and in this connection we should like to see further investigations given.

In Chapter VII. the author discusses series of functions, their continuity and their derivates. He gives $\sum_n \frac{\sin(n!\,2\pi x)}{n!\,2\pi}$ as a continuous function which has a derivate for no value of x. Chapter IX. is concerned with power-series. In the succeeding chapter we have a somewhat fragmentary discussion of integrals and primitive functions followed by an investigation of singular integrals and Fourier's series. Here we expect and find some curious results. *Non-zero functions exist of which the integral vanishes for every value* x *comprised in the interval where the functions are defined.* And again, *every continuous function has an infinity of primitive functions, and only those primitive functions which are uniformly differentiable belong to the set* $\int_{x_0}^{x} f(x)\,dx+\mathit{const}.$

The final chapter of the volume gives scattered notions and theorems in the domain of functions of two variables. It is the natural extension of the previous ideas.

A somewhat long account of M. Robin's book has been given because of the undoubted interest of its novel point of view. We must regret Robin's early death and the unfinished state of his work: perhaps M. Raffy will do more than is comprised in the present volume to complete his friend's investigations. It is very doubtful whether the new theory will ever displace the traditional point of view: it deserves, however, to be read in that the very

strangeness of its propositions may make the reader consider again the difficult ideas which underlie the theory of functions of a real variable and the theory of number on which it is based.

E. W. BARNES.

An Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups. By J. E. CAMPBELL, M.A. Pp. xx, 416. 1903. (Clarendon Press.)

This book is properly, though modestly, called introductory, because it starts *ab initio*. It is, however, comprehensive. Any pretence to be exhaustive it disclaims: but it goes a long way, and is very thorough, never shirking difficulty or sacrificing generality in the supposed interest of the beginner. It is, of course, on *Lie's* theory of continuous groups: there is no other theory of them. But its presentation of the theory is by no means borrowed from any previous work.

Sophus Lie was a genius. A thinker of marvellous power, vast energy, unrestrained egoism, he was inspired with a new grand idea, saw far as to whither it could lead, laboured enthusiastically, and was always at the forefront with it, never content that any other should be alongside except as almost part of himself. In these modern times, when so many skilful and highly trained mathematicians are ever keen to discover an opening for fruitful research, it testifies strongly to the pre-eminence of the man, that the highly developed theory which has resulted from his initiative should still owe relatively (though not absolutely) little to workers not directly associated with himself. Such workers are busy, however; and now, alas! there is no Lie left to forge ahead of them.

Great works exist on the elaborated theory. Their authors produced them with and for Lie himself. The time has come for treatises without the impress of direct guidance from the master, written by his independent followers. Of these Mr. Campbell is one. He has saturated himself with Lie's ideas, has thought them out again for himself, has done good work in investigations connected with them, and now gives us the benefit of seeing how the train of them looks most natural to him. It is not only to the English reader that he thus does a service. The book will be a standard work elsewhere, as the first considerable one on the subject regarded as classical and not as the property of its originator. But to the English reader his service is specially great. A translation of Lie-Engel or Lie-Scheffers would, though ponderous, have been acceptable as removing merely linguistic barriers in the way of our familiarization with the study. We wanted more, however, a book neither too short nor too expanded, one from a point of view accessible to those brought up as we have been, one with the kind of condensation which we appreciate, with the insular traits to which we are accustomed, and with shortcomings, if any, such as we know how to discount. And we now have a good one.

It is needless in this notice to expatiate on what transformation-group theory is. Earlier reviews in the *Gazette* have done so. Mr. Campbell begins with two excellent introductory chapters devoted to making the matter clear. They will be most valuable as furnishing ideas, even to those who have no intention of striving to master much of the elaborate analysis connected with later developments. They flow easily when the slightly too condensed first two pages are passed. Then follow three powerful chapters on the infinitesimal operators of a group, and the three fundamental laws as to their relationship and combination. Much here is unavoidably difficult. The treatment has considerable novelty. The proof of the converses of the second and third laws is briefer, and more readily convinces, than other proofs which have been elaborated. Much use is made of an "exponential" theorem (§50), which is an important one due to the author himself (later but independently also arrived at by Pascal). At an earlier point (§40) the all-important first law is proved with elegance, but unfortunately stated with some obscurity owing to excessive brevity. The reader must look on twenty-eight pages to §59 for help to realize it, and yet fourteen pages more to §67 to find an explicit statement of what it entails. This is almost a unique case in the book of lack of helpful explanation at the outset. The author's ordinary practice is the admirable one of preparing the student to realize the object of a difficult piece of analysis which is coming,

by a few well-chosen words in advance descriptive of the aim and purport of the investigation.

After these chapters follow a good short one on complete systems of linear differential operators, an adequate discussion of the admission of known groups by differential equations, and a very interesting chapter on invariant theories. Primitivity, similarity, isomorphism, etc., of groups follow: then the theory of the construction of groups with known structure constants. Everything is at once presented with full generality: not led up to from the particular. The particular is only given by way of example. Pfaff's equation and Pfaffian systems next appear, and the author is led on to another subject, on which he speaks with special force and interest: that of contact transformations. This he at length leaves to describe, in chapters xxi. to xxiv., how all possible groups in not more than three variables can be obtained. Of these four chapters the first offers attractive reading, even to those to whom the three dimensional problem is repellent from its tediousness. A final chapter, with a note acknowledging indebtedness to Scheffers, but containing also some novelty, is devoted to certain linear groups connected with higher complex numbers.

The style throughout is terse and direct. Misprints are neither numerous nor absent. The printing is very clear: indeed, the Oxford Press is to be congratulated on a success in making suffixes legible, which is very unusual, and is of great value in a book on a subject like this, which requires them to be many, and generally triple. There is a good index and a very full and useful table of contents.

At times one would have liked the examples to be in a different type, or indented in the text. As a general rule, division into short articles has removed the danger of obscurity; but there are exceptions. The lengthy § 44, for instance, contains many things; and had it been clear at a glance where one thing ended and another began, the reader would have been helped.

E. B. Elliott.

Elements of the Theory of Integers. By J. Bowden, Ph.D. Pp. x, 258. (Macmillan.)

The reader of this book is at once struck by the fact that the author writes "ar," "siv," and "thot" for "are," "sieve," and "thought" respectively, while retaining the usual spelling of "one" and "two." The same spirit of unsatisfactory compromise unfortunately pervades his analysis, and it is difficult to see what good purpose is served by the publication of a work of this kind. It is too abstract for the ordinary reader, and not nearly logical enough to satisfy a specialist. Professor Bowden appears to have consulted his authorities without being able to estimate their relative importance, and he has not taken up any consistent attitude of his own. He makes no reference to Dedekind's *Was sind und was sollen die Zahlen?*, and is presumably unaware of its existence; otherwise he would hardly have made $b+a=a+b$ an axiom. The fact that he does this, while spending two pages to prove that the sum of two even integers or of two odd integers is even, while the sum of two integers, one of which is even and the other odd, is odd, illustrates his vague attitude and lack of the sense of proportion. It may be added that counting is discussed after propounding this so-called axiom, the only justification given for which is derived from collecting two groups of objects into one. To add to the confusion, we are told in a foot-note that "the idea of number is prerequisit [*sic*] to, not derived from, the idea of counting": and this after saying in Art. 1 that "to every group of objects belongs a number, the number of objects in the group"; a statement followed by numerous propositions, with proofs based upon the unanalysed, and certainly not primitive conception of "a number [*Anzahl*] of objects." Contrast with this the conscientious minuteness of the following:

"In symbols $a=a$.

"A longer statement of the same principle is: *The number of objects in a group is the same as the number of objects in that group.*"

It is only fair to say that some parts of the book, such as the chapter on zero and negative integers, are comparatively satisfactory; but it contains nothing which has not been done, and done better, before.

G. B. Mathews.

Geometrie der Dynamen. Die Zusammensetzung von Kräften, und verwandte Gegenstände der Geometrie. Von E. STUDY. (Leipzig, Teubner, 1903.) Pp. 603. M. 21.

The first half of this book was published in 1901, and consists of two parts treating of the composition of forces and wrenches, first by geometrical and secondly by analytical methods. The second half and third part was published last year, and contains a new sort of line geometry which the author has invented, with numerous and far-reaching applications.

These pages contain a kernel of sterling value concealed within an immense husk of almost impenetrable material which would prove trying to even the strongest intellectual teeth; but the kernel, when once reached, is one of those pieces of mathematics which give a thrill of satisfaction to all who appreciate them, by their beauty and simplicity in addition to their enormous power.

This principle, which was explicitly stated by Mr. Homersham Cox twenty-two years ago, occurs at the end of Prof. Study's second part. Many of the theorems presented geometrically in the first part are deductions from it, and are consequently almost unintelligible where they occur. The author takes the greatest care to prepare for every possible exception or particular case of a metrical nature, and this leads him to invent a bewildering number of new names, many of which connote overlapping classes of things. Thus, for example, two lines not in the same plane form an "impulsor," and two lines which are not perpendicular nor at infinity form a "motor," and so on. It is doubtful whether the passion for exact definition which leads to such a picking to pieces of the subject tends to any true advance: it certainly checks the progress of the reader. The third part of the book is concerned with a new set of six line-coordinates designed to overcome difficulties arising at infinity; of this part, containing doubtless much important matter, no notice is taken here, but the reader is referred to an enthusiastic review in the January number of the *American Bulletin*.

The mathematical principle which led to the writing of this book is best presented in the first instance as a simple device whereby much of the labour of calculating with line coordinates may be avoided, and known results may be extended from two dimensions to three. A line through a fixed point (the origin) is determined in reference to a given frame by three direction cosines, l, m, n, satisfying the single relation

$$l^2+m^2+n^2=1,$$

and any line in space is determined by six coordinates l, m, n, l', m', n' satisfying the preceding relation and one other, namely

$$ll'+mm'+nn'=0.$$

It follows from this that if l, m, n are regarded as functions of a parameter, their differential coefficients *may* be regarded as the second set of three coordinates of a line in space, and further, of *any* line having the direction l, m, n. Anyone can follow out this idea for himself; it leads to the rule: by differentiating the solution of a problem concerning lines through the origin the solution is obtained of the corresponding problem for lines in space. For example, the last three coordinates of the common normal of two lines are deduced by simply differentiating its direction cosines. Again, the well known theorem that two polar reciprocal triangles are in perspective leads to the beautiful theorem, discovered independently by Prof. Morley of Baltimore and Dr. Joh. Petersen of Copenhagen in 1898, that the three common normals of opposite sides of a skew rectangular hexagon have a common normal. Conversely, Study's "Übertragungsprincip" and Petersen's "Principe de duplication" can be based in a purely geometrical way upon this theorem proved independently, or on the equivalent theorem, employed by the latter writer, that simultaneous increments of the sides and angles of a spherical triangle are proportional to the sides of a certain skew right-angled hexagon. For more detailed explanations and exact references the reader is referred to two papers on this subject in the *Messenger of Mathematics* of 1902.

But this principle is not merely an ingenious device. It lies at the foundation of all line geometry, including that of "elliptic" and "hyperbolic" space. The three types of *absolute* can be written in one form,

$$\epsilon^2(x^2+y^2+z^2)+t^2=0,$$

where $\epsilon^2=-1$ for hyperbolic, $\epsilon^2=0$ for Euclidean, and $\epsilon^2=1$ for elliptic geometry. The "rank equation," upon which the metrical relations of lines depend, is

$$l^2+m^2+n^2+\epsilon^2(l'^2+m'^2+n'^2)=0.$$

This can be written in the form

$$(l+\epsilon l')^2+(m+\epsilon m')^2+(n+\epsilon n')^2=0,$$

resembling the absolute of spherical geometry. In this lies the ultimate reason for the existence of the principle of extension.

In these equations ϵ is a *number*, real or imaginary. The next step in logical sequence takes its origin in one of Clifford's suggestions, to whom so many of the ideas in this region can be traced back. He introduced a *symbol* ω, obeying the ordinary laws of algebra, and in addition the equation $\omega^2=1$. With this he coupled two vectors together to form a "bivector" and two quaternions to form a "biquaternion." Using the same symbol, the *dual coordinates* of a line (l, m, n, l', m', n') are defined in the first instance to be $l+\omega\epsilon l'$, $m+\omega\epsilon m'$, $n+\omega\epsilon n'$, and it is then evident that $\omega\epsilon$ may be replaced by a single symbol obeying the same laws as ϵ in the three cases. In hyperbolic and Euclidean geometry ω may be omitted without confusion: in elliptic geometry ϵ is omitted, being unity, and then the presence of ω distinguishes the "dual coordinates" of a line from the "Clifford parallels." This principle was first clearly formulated in 1882 by Mr. Homersham Cox in his paper "On the Application of Quaternions and Grassmann's Ausdehnungslehre to different kinds of Uniform Space" in the *Cambridge Philosophical Transactions*, vol. xiii., p. 69. On p. 112 he says, "We see then that in all three kinds of geometry the properties of lines in space can be derived from those of lines meeting in a point if we write . . . $l+\omega l'$, etc., for its direction cosines . . . where in the first kind of geometry $\omega^2=-1$, in the second $\omega^2=0$, and in the third $\omega^2=1$."

With this quotation in mind, it is interesting to read, on the last page of the book, how inadequately certain English mathematicians have attempted to expound Clifford's ideas. They are regarded as "lamentable sacrifices to a system of education which is still in vogue in England"! The germ of truth in the last phrase serves only to accentuate its inappropriateness in connection with the work of Cayley, Hamilton, Clifford, and others. The subject of Universal Algebra, with all its geometrical interpretations, received much of its development in this country, and while assistance is welcomed from all quarters, it should not be dealt out with a too heavy hand. R. W. H. T. HUDSON.

UNIVERSITY OF LIVERPOOL.
February, 1904.

Technical Mechanics. By E. R. MAURER, Professor of Mechanics in the University of Wisconsin. Pp. vi, 382. Price 17s. net. 1903. (New York: J. Wiley & Sons. London: Chapman & Hall.)

Professor Maurer's work may be regarded as intermediate between an elementary text-book on the one hand, and the standard treatises on special branches of mechanics on the other. Presupposing in the reader a certain amount of elementary knowledge, the author discusses statics, some simple cases of attractions, and the plane motion of a particle and of a rigid body.

In statics, graphical and analytical methods of proceeding are developed side by side. The graphical treatment of the conditions of equilibrium of forces in one plane, and of forces in space, may be alluded to as specially noteworthy.

Professor Maurer has indicated the correct aim of a book on Technical Mechanics in his preface in the following sentence: "On the theoretical side practically all the subjects treated have a direct bearing on engineering problems. On the applied side no attempt is made to present fully any one subject, such as . . . the balancing of rotating systems, for the object in view was the illustration of the use of the principles of mechanics, and not treatments of . . . balancing. . . ."

In carrying out this design much skill and judgment is shown in the selection of illustrative examples of practical interest, without allowing the question of principle under discussion to be obscured by mere details. Minute practical details in a text-book are apt to be the details of the practice of yesterday, not to say of the day before.

Unfortunately, the treatment of particular topics is not always maintained at the same level of excellence as the planning of the broad outlines. The following extract (page 190) illustrates my contention that a student, and not necessarily, I think, a merely careless or hasty student, might easily obtain erroneous notions from several paragraphs of the work.

In investigating expressions for the tangential and normal accelerations of a particle moving on a plane curve, the author takes the axes of x and y parallel to the tangent and normal at P, and continues "*with axes thus chosen* (author's italics) it is plain that the tangential and x components and the normal and y components are equal, or

$$a_t = a_x = \frac{dv_x}{dt}, \quad a_n = a_t \text{ (sic)} = \frac{d^2y}{dt^2}.$$

Now
$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\frac{dx^2}{dt^2} = \frac{d^2y}{dx^2}v_x^2;$$

also $\frac{d^2y}{dx^2}$ equals the curvature of the path at P. Denoting the radius of curvature at P by ρ, and since $v_x = v_t = v$,

$$a_t = \frac{dv}{dt} = \frac{d^2s}{dt^2} \text{ and } a_n = \frac{v^2}{\rho}."$$

The statements in the penultimate sentence quoted happen to be true because $\frac{dy}{dx} = 0$ at P, but the coincidence cannot be considered as equivalent to a proof. In fact, the reasoning would equally serve to prove $a_n = v^2\frac{d^2y}{dx^2} \times f$, where f is any function of $\frac{dy}{dx}$, which becomes unity when $\frac{dy}{dx} = 0$.

Notwithstanding a certain absence of precision in expression, particularly as regards kinetics, of which the foregoing extract gives an unfavourable instance, the excellently chosen examples, the good and clear printing, and the conciseness of the treatment make the book decidedly an attractive one.

Professor Maurer, it is interesting to observe, adopts the "engineers'" units and proposes the terms "geepound" and "geekilogram" for the "engineers'" unit of mass.

On page 254, Ex. 5, $a^2/3$ should be $a^2/12$, and on page 280 $m\bar{r}\omega$ should be $m\bar{r}\omega^2$.

C. S. Jackson.

An Elementary Treatise on Cubic and Quartic Curves. By A. B. Basset. Pp. xvi + 255. 10s. 6d. 1901. (Deighton Bell.)

The author tells us in the preface that "the present work originated in certain notes, made about twenty-five years ago," and the general effect is one of discontinuity. The earlier chapters of the book mainly reproduce some of the results contained in the first half of Salmon's *Higher Plane Curves*; the methods of obtaining these results are novel. The treatment of Plücker's equations is specially striking, and equally unsatisfactory; the author would, in fact, have done better to have omitted his first four chapters and given a reference to Salmon's work in the preface.

Mr. Basset confesses to a liking for the use of anglicised Greek words, and remarks that mathematicians would do well to follow the example of the medical profession, and, when they require a new word, have recourse to the Greek language. Acting on this principle, he introduces into his text words such as autotomic, anautotomic, endo-dromic, exodromic, perigraphic, etc., etc. The second half of the book is a great improvement on the first: curves of the third order and of the third class respectively are duly classified (without, however, any reference to recent work in this direction) and some space is devoted to the special treatment of historical curves like the trisectrix, the cissoid of Diocles, Descartes' folium, and the witch of Agnesi.

Since the book is of a professedly elementary character, the classification of quartic curves, by means of the lines on a cubic surface and the use of Schläfli's notation, is not to be expected; the classification given is therefore naturally

somewhat loose. In the later chapters the projection of tricircular quartics is discussed.

The whole question of plane cubics and quartics is nowhere seriously undertaken; elliptic functions are not used, and some of the analysis is of a very clumsy character. The subject matter is so diffuse that a good index is indispensable to anybody using the book as a text-book: but the book cannot be recommended as a text-book, so perhaps the absence of an index is immaterial. The appearance of this treatise once again emphasises the necessity for a new edition of Salmon's *Higher Plane Curves*, brought up to date with the researches of the past twenty years. Now that the learned author is no more, perhaps some enterprising geometer will undertake the task: practically, the only modern work on the subject is Fiedler's German translation of Salmon, and a comparison of the contents of the original work and its translation is sufficient to show how far the subject has grown in the last few years. P. WORSLEY WOOD.

Lectures on the Logic of Arithmetic. By M. E. BOOLE. (Clarendon Press.) While disagreeing with some of the doctrines laid down in this book (and humbly acknowledging that we may be quite wrong in doing so), we can yet heartily recommend it to all those who are anxious to gather in from whatever source any ideas which may add to the efficiency of their teaching. Many of us who are "too far in years to be a pupil now" to the philosophy of the authoress, will rejoice to see that she reaches the same results by highly philosophical reasoning that we have found out by the experience of the class-room. We advise all to read on past whatever they are inclined to disagree with, and come to the practical bearing of the precepts laid down. The "mental picture" method is of wide application. The illustration of the nature of curves by the study of the dog following the man, the nature of the minus sign by the nature of "change" in shops are cases in point. If any teacher of higher forms should be irritated by the childish way in which the dialogue is put, he may easily *mutatis mutandis* adopt the method used.

A New Geometry for Junior Forms. By S. BARNARD, M.A., and J. M. CHILD, B.A. (Macmillan & Co. 2s. 6d.) Among the additional sections which have been incorporated with the easier parts of the New Geometry to form the present volume, are a set of Introductory Exercises on Experimental work, an explanation of the Construction of Scales and a well illustrated account of the Forms of *cube, cuboid, tetrahedron, pyramid, prism, cylinder and cone.* These, and especially the last, add much to the usefulness of the book. It was a wise step of the Universities to include the forms of the simpler solids in their elementary syllabus. Under the old method, a pupil who left without reading XI. 1-21, stood a chance of never receiving any systematic instruction in the simpler solid forms. From our own experience, we believe few things so well calculated to arouse attention and sustain interest than the inspection and construction of models. The authors give the *nets* as well as the perspective views, and suggest the construction of others of varied dimensions for solids whose form is given in perspective and dimensions in words. Such exercises are beneficial, and from these the pupil may easily advance to easy orthogonal projection for representing *in plano* the solid he has planned and constructed.

Elementary Geometry. By CECIL HAWKINS, M.A. (Blackie & Son.) A useful treatise on the subject matter of Euclid I.-IV., in which the author seems to be inspired by a wish to go as far as possible in the direction of the reformers, and has no wish to make his treatise look like Euclid brought up to date. We are introduced to symmetry (axial and central) and images. There is an extensive and varied collection of exercises both in Theory and Practice.

Elementary Geometry. Section III. By F. R. BARRELL, M.A., B.Sc. 1s. 6d. (Longmans, Green & Co.)

This section deals with the subject matter of Euclid's Book XI., and gives, in addition, a careful treatment of the Cylinder, Cone, and Sphere. It can be heartily recommended. We have drawn attention, in receiving the previous sections, to the great care and skill shown in the diagrams. These qualities are still more marked in the third section, as Solid Geometry gives more scope for their exercise. Plato's five regular solids get very scant attention, being

merely noticed in some exercises, but the space gained by omitting them is perhaps as well, or more usefully occupied by the Sphere, which deserves to have its properties made part of every school course of Solid Geometry. Areas and Volumes are satisfactorily treated.

An Introduction to the Study of Geometry. By A. J. PRESSLAND, M.A. 1s. (Rivingtons.)

In this little book we find the same excellence in execution of diagrams and arrangement of matter as in the author's *Geometrical Drawing*, a work which advocated experimental treatment when it was not so fashionable as it is now. The theorems selected for experimental treatment are calculated to arouse the interest of the young student, and to lead him to try further mathematical experiments on his own account. There is a useful introduction to Graphical Arithmetic.

Dr. E. Bardey's Anleitung zur Auflösung engekleideter algebraischer Aufgaben. By F. PIETZKER. (Teubner.)

This seems a new edition of an old work of which little remains except the title. It aims at showing how problems wrapped up in ordinary language are to be translated into that of Algebra. The examples are taken from widely different sources, Trigonometry, Astronomy, Mechanics, and Physical Science.

Scale and Protractor for Blackboard Drawing. Designed by A. J. PRESSLAND, M.A. Post free, 2s. 6d. (Made and sold by W. Brunton, West Silverwells, Edinburgh.)

This serviceable instrument has three scales marked on it. On one face degrees at intervals of 10°, and a line scale extending on each side of the zero to 14 inches at intervals of one inch. On the other face it is graduated to 35 centimetres on each side of the zero at intervals of 5 centimetres. We have tested the accuracy and found the graduations quite fine enough for board work, at the same time that the stoutness of its make enables it to stand the wear and tear of a school instrument. We find it very useful. E. M. LANGLEY.

A Course of Pure Geometry. By E. H. ASKWITH. Pp. xii, 208. 1903. (Cambridge Univ. Press.)

This book consists of a preparatory course of lessons for such students as wish to specialise in Pure Geometry, as well as for those who merely require some general inkling of the modern methods in that subject. The author acknowledges his indebtedness to Messrs. Casey, Lachlan, and J. W. Russell, as, indeed, must any writer on Pure Geometry for some time to come. But the course of lessons shews no slavish adherence to the methods of any other author, and in a measure may be said to mark an advance from the pedagogic point of view. For Dr. Askwith is careful to point out that the distinction between pure and analytical geometry "is one of method rather than of idea," and an occasional digression to view the matter of a section from the analytic standpoint is no doubt fruitful in result. There can be no real divorce between the two when we admit the doctrine of continuity. We have no doubt that this text-book, with its excellent collection of questions from Tripos and other papers, will be found to "fill a gap."

Annuaire pour l'An 1904. Published by the Bureau des Longitudes. Pp. 732, 116. 1 fr. 50 c. 1903. (Gauthier-Villars.)

Last year we drew attention to the new arrangements under which the *Annuaire* is published. To avoid "vain repetition," the usual tables are printed in such order that to have the complete set the reader must possess the *Annuaire* for two consecutive years. For instance, the present number contains in detail all the tables relative to Chemistry and Physics, and does not contain those referring to geographical and statistical data. The chemical and physical tables will not be given in the number for 1905, and will be replaced by those dealing with Metrology, Money, Statistics, and so on. So again, this year's *Annuaire* is *par excellence* the number for ancient calendars, and for a complete table of the elements of the

minor planets. Next year it will be the turn of stellar parallaxes, double stars, stellar spectroscopy, etc. The essays, which have long been a characteristic feature of this publication, deal this year with the International Geodetic Conference, Aug. 1903, by M. Bouquet de La Grye, and an elementary explanation of the Theory of the Tides, by M. P. Hatt. We again call attention to the 848 pages of tables and letterpress for one franc fifty cents.

Arithmetical Types and Examples. By W. G. Borchardt. Pp. xii, 367. 1903. (Rivingtons.)

This is one of the first Arithmetics to endeavour to follow up the recommendations of the Committee of the Mathematical Association, and is an eminently successful exposition of the subject on the improved lines. The greater number of the examples are original, and those in the Test Papers are from papers set in public examinations. The exercises are each preceded by a model solution, with notes where needed; and another edition is to be published for the benefit of classes under teachers who object—as perhaps most of us do—to worked-out examples.

School Algebra. By J. M. Colaw and J. K. Ellwood. Pp. 432. 1903. (Johnson, Richmond, Va.)

This is an excellent introduction to the elements of Algebra, and as a special feature it develops from the outset the arithmetical basis of the subject. "Formulae and results are frequently tested by arithmetical applications, and in passing to new ideas numerical illustrations are often used." The Binomial and Remainder Theorems are early introduced—the former in the chapter on multiplication, the formal proof being of course deferred. Graphs are freely used throughout, and the questions set at the end of the sections are as far as possible drawn from the experiences of everyday life. A thoroughly careful little introduction to Algebra.

A Manual of Practical Mathematics. By F. Castle. Pp. vii, 541. 1903. (Macmillan.)

This is the necessary sequel to Mr. Castle's previous volumes on Practical Mathematics, and brings the elements of the "higher" Mathematics within the grasp of the ordinary student. The treatment throughout is clear, and in every way justifies the author's hopes that the average student will be enabled to apply to practical purposes in a short time the principles of those portions of advanced mathematics of which he otherwise would long remain in ignorance. Among the sections in the book, for which Mr. Castle, rightly or wrongly, claims the attribute of novelty, are the following: The graphic solution of quadratic equations—the identification of a plotted curve by celluloid strips on which the standard curves are displayed—solution of equations of type $T=a+by^n$—graphical treatment of $y=a\cos(\omega t+e)$, $y=a\sin(bx+c)$—Amsler planimeter, Fourier's theorem—obtaining slope of curve by set square and pencil—geometrical proof that

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$$

—the use of progressions to illustrate the Integral Calculus. We are inclined to think that this book will meet with a large sale.

The Schoolmasters' Year Book and Directory. 1904. Pp. lx, 1030. 5/- net. (Swan Sonnenschein.)

The second issue of the Year Book is larger than its predecessor, mainly owing to an additional fifteen hundred names in the Directory. The editor complains that incorrect information is not uncommon. The date of the foundation of certain schools were given as 1945, 1904, 184, and so on. There is a full list of the Education Committee of County and County Borough Councils. The publishers are to be congratulated on the success which has attended the appearance of this "excellent and indispensable" book of reference, and the editor on the absence of errors—as far as we have been able to test it.

Vorlesungen über Algebra. By Dr. G. Bauer. Pp. vi, 373. 1903. (Teubner.)

This volume was printed at the instance of the *Mathematischer Verein* of Munich as a compliment to Professor Bauer on the occasion of his 80th birthday, in November, 1900. From the portrait of the author, which forms the frontispiece, we gather that in spite of his advanced years his intellectual faculties are not yet dimmed. The lectures are those given to students in their first and second year, and cover a wide range of subjects. The first section contains the elementary properties of polynomials and rational fractions, the theory of symmetrical functions, elimination and the transformation of equations. On p. 13 we have a construction for $z=x+yi$ by means of similar triangles, where n is needlessly restricted to positive integers. The fundamental theorem that every algebraical equation has a root is proved in much the same way as in Chrystal, and is exposed to the same objection. If for every value of $f(z)$ an h can be found such that mod $f(z+h)$ is less than mod $f(z)$, then $f(z)$ does not necessarily vanish. For example, Moritz gives $f(z)=\left(\frac{1+z^2}{z^2}\right)^{1+z^2}$, where z has no zero value, and yet for every value of z an h may be found such that $f(z+h)$ is less than $f(z)$.

The second part deals with the algebraical solution of equations, the elements of the Galois theory, binomial equations, Abelian equations, and a few properties of numbers. Part III. treats of the numerical solution of equations, the system of Graeffe as improved by Encke being that preferred by the author. The principle is simple enough. From the given equation a second is derived, whose roots are very high powers of the roots of the original equation. The powers of the smaller roots may then be neglected in comparison with the powers of the larger roots. The volume concludes with the theory of determinants and their applications, and two notes on continued fractions, and Lagrange's formula for obtaining the sum of the nth powers of the roots of a quadratic. The book is clear and is easy to read. Many beginners will find it exactly suited to their needs.

I Tre Problemi Classici degli Antichi: studio storico-critico. Problema Secondo:—La Duplicatura del Cubo. By Prof. Bellino Carrara, S.J. Pp. 114. 1903. (Fusi, Pavia.)

Professor Carrara has carried his researches into the three famous problems of antiquity one stage further by the publication of the section dealing with the Delian problem—the duplication of the cube. The first ninety pages do not contain much that is not easily accessible to English readers in such books as Rouse Ball's *Recreations* and *Short History*, Heath's *Apollonius of Perga*, Allman's *Greek Geometry*, etc. But in the sections dealing with the problem as treated by the mathematicians of the seventeenth to the nineteenth centuries we find much that is novel and interesting. Here, curiously enough, Professor Carrara seems to be unaware of the part played by the English philosopher of Malmesbury in the course of his deadly feud with the mathematician Wallis. For years Hobbes and Wallis had been disputing in terms as unseemly as they were biting, about the squaring of the circle, until Wallis gave Hobbes up in despair. The latter tried in vain to induce Wallis to renew the argument. He then adopted wily tactics. He published in Paris under an assumed name what he fondly believed to be a solution of the famous Delian mystery. It was in French, and his simple artifice succeeded, for Wallis came out of his lair and demolished the so-called proof. Hobbes immediately claimed the authorship, defended it, making more blunders than ever in the process, and re-published it in his *Dialogus Physicus, sive de Natura Aeris*—a vehement attack on Boyle and other friends of Wallis. These he hated, for in the new Royal Society which they had just been instrumental in forming they had maliciously left out the redoubtable philosopher. Hobbes once more was crushed by Wallis in the scathing *Hobbius Timorumenos*, and was silent for some years. But when he was over eighty years of age he published his *Quadratura Circuli, Cubatio Sphaerae, Duplicatio Cubi.* Once more the tireless Wallis refuted the pretensions of Hobbes, but the latter, nothing daunted, re-published his book, with answers to all objections, and the volume was dedicated, Professor Carrara will be

interested to hear, to the Grand Duke of Tuscany. The battle went on for some years, and Hobbes fired his last shot at Wallis at the age of ninety! Those of our readers who ever come across the effusions of Wallis in this squabble will find that they are extremely rare. Professor Carrara's eighth chapter deals with the work of Mascheroni, Vargiù, Bonafolce, and Boccali in connection with approximations to $\sqrt[3]{2}$, etc. A few remarks on the use of the integraph bring an interesting brochure to a close. In one point the author is "sadly to seek." His quotations from French and German, even when only the titles of books, are often marred by inaccuracies. Thus we have "algebreischen" (p. 9); three mistakes in a quotation on p. 12; four in the title of Chasles' Aperçu (p. 18); commas have replaced dashes in the solution (p. 19); Allman, Greek Geometrie (p. 25); the same book figures as Greeck Geometrie (p. 32); Tannery's Histoire general (p. 33); Archimcdis (p. 37); perga for Perga (p. 39); contemporaim (p. 42); goschichte (p. 43); wisseuschaften (p. 48); and so on. Even Apollonius is treated to an extra "p." It is hardly an exaggeration to say that thirty per cent. of the French accents are wrongly placed. This careless correcting of proofs mars what is otherwise a sound piece of work.

NOTICE.

THIRD INTERNATIONAL CONGRESS OF MATHEMATICIANS AT HEIDELBERG, AUGUST, 1904.

Members of the Mathematical Association are invited to be present at the Third International Congress of Mathematicians, which will take place at Heidelberg from Monday, August 8th, to Saturday, August 13th, 1904. The proceedings will commence with a reception at 8 P.M. on August 8th; there will be three general meetings, and also meetings of six sections, viz., (1) Arithmetic and Algebra, (2) Analysis, (3) Geometry, (4) Applied Mathematics, (5) History of Mathematics, (6) Pedagogy.

The centenary of the birth of C. G. J. Jacobi will be celebrated during the Congress.

There will be an exhibition of Mathematical Models (historical and modern), and an exhibition of the more important mathematical publications during the past ten years, to which contributions on loan are invited. Intending contributors to the exhibition of literature are requested to correspond with Dr. A. Gutzmer, Professor an der Universität Jena. Social meetings, a banquet, and an illumination of the Castle will also take place.

The price of a ticket for the Congress is 20 marks—additional tickets may be obtained by members of the Congress for their own party at 10 marks each person, which are available for the general meetings and festivities.

Further information concerning the Congress may be had from Professor A. Krazer, Karlsruhe i. B. Westendstrasse, 57.

ERRATA.

No. 43, p. 383: lines 13, 14. For $P\hat{T}R$ read $E\hat{T}R$.

,, $P\hat{Q}R$ read $A\hat{Q}R$.

Dele. and $R\hat{F}A = A\hat{Q}R$.

Front Cover, Jan. 1904. For Vol. III. read Vol. II.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) **For Sale.**

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and Published by E. HENDRICKS, M.A., Des Moines, Iowa, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

Proceedings of the London Mathematical Society. First series, complete. Vols. 1-35. Bound in 27 vols. Half calf. £25.

Cayley's Mathematical Works. Complete, equal to new, £10. Apply, Professor of Mathematics, University College, Bangor.

(2) **Wanted.**

Vols. I.-IV. *Mathésis.* 1881-1884.

The Messenger of Mathematics. Vols. 2, 15-20, 24, 25.

Tortolini's Annali. Vol. I. (1850), or any one of the first eight parts of the Volume.

Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

(3) Dr. Muir, The Education Office, Cape Town, will give Vol. 109, *Crelle's Journal*, to any member of the Mathematical Association whose set is without it.

BOOKS, ETC., RECEIVED.

Geometric Construction of the Regular Decagon and Pentagon Inscribed in a Circle. By H. H. LUDLOW. Pp. 12. 1904. (Open Court Co., Chicago.)

Experimental and Theoretical Course of Geometry. By A. T. WARREN. Pp. viii, 264. 2s. 2nd edition. 1904. (Clarendon Press.)

Graphs and Imaginaries. By J. G. HAMILTON and F. KETTLE. Pp. 41. 1s. 6d. 1904. (Arnold.)

Lehrbuch der Analytischen Geometric. Erster Teil. By. O. FORT. 7th edition. pp. xvii, 268. 1904. 4 m. (Teubner.)

Anschauliche Grundlagen der Mathematischen Erdkunde. By K. GEISSLER. Pp. vi, 199. 1904. 3 m. (Teubner.)

Handbuch der Theorie der Cylinderfunctionen. By N. NIELSEN. Pp. xii, 408. 1904. 14 m. (Teubner.)

A School Geometry. Parts I.-IV. H. S. HALL and F. H. STEVENS. pp. xii, 340, x. 4/6. 1903 (Macmillan). (Substance of Euclid Books I.-IV. and VI. with Additional Theorems and Examples.)

A School Geometry. Parts I.-V. (Containing the substance of Euclid, Books I.-IV., VI., with Additional Theorems and Examples.) By H. S. HALL and F. H. STEVENS. pp. xii, 340, ix. 4/6. 1904 (Macmillan).

Elementary Geometry. Section III. By F. R. BARRELL. Containing the subject matter of Euclid Book XI., together with a full treatment of Volume and Surface of the Cylinder, Cone, Sphere, etc. pp. 285-360. 1/6. 1904 (Longmans).

On the Transfinite Cardinal Numbers of Well-ordered Aggregates. By P. E. B. JOURDAIN. pp. 61-75. (Phil. Mag., Jan. 1904.)

Introduction to Quaternions. By the late Professors P. KELLAND and P. G. TAIT. 3rd Edition. Prepared by C. G. KNOTT. pp. xvii, 208. 7/6. 1904 (Macmillan).

First Lessons in Observational Geometry. By Mrs. W. N. SHAW. pp. ix, 148. 2/. 1904 (Longmans).

Geometry on Modern Lines. By E. S. BOULTON. pp. viii, 126. 2/. 1904 (Methuen).

Elementary Geometry. By W. M. BAKER and A. A. BOURNE. pp. xxix, 477. Third Edition. 4/6. 1904 (Bell).

A Key to Elementary Geometry. By W. M. BAKER and A. A. BOURNE. pp. 177. 6/. 1904 (Bell).

Analytische Geometrie der Kegelschnitte. By G. SALMON. Edited by W. FIEDLER. Part II. 6th Edition. pp. xxiv, 443-854. 1903 (Teubner).

Sitzungsberichte der Berliner Mathematischen Gesellschaft. pp. 68. 1903 (Teubner).

Der Geometrische Vorkursus in Schulgemäszer Darstellung. Edited by E. WIENECKE. pp. 97. 1904 (Teubner).

Jornal de Sciencias Mathematicas e Astronomicas. Edited by F. GOMES TEIXEIRA. Vol. xv. No. 3. 1903 (Univ. of Coimbra).

Aufgaben aus der niederen Geometrie. By I. ALEXANDROFF. pp. vi, 123. 1903 (Teubner).

A New Geometry for Junior Forms. By S. BARNARD and J. M. CHILD. pp. xvi, 306. 2/6. 1904 (Macmillan).

Five-Figure Tables of Mathematical Functions. By J. B. DALE. pp. xv, 92. 3/6 net. 1903 (Arnold).

Theoretical Geometry for Beginners. Part III. By C. H. ALLCOCK. pp. 113. 1/6. 1904 (Macmillan).

Exercices Méthodiques de Calcul Intégral. By E. BRAHY. New Edition. pp. vi, 301. 1903 (Gauthier-Villars).

Algèbre Supérieure. Parts I. and II. By C. DE COMBEROUSSE. pp. xxi, 767; xxiv, 831. 1890-1904 (Gauthier-Villars).

Cours de Mathématiques Supérieures. By L'ABBÉ STOFFAES. 2nd Edition. pp. vii, 537. 1904 (Gauthier-Villars).

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